Multipole expansion of acoustical Bessel beams with arbitrary order and location

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Abstract: An exact solution of expansion coefficients for a T-matrix method interacting with acoustic scattering of arbitrary order Bessel beams from an obstacle of arbitrary location is derived analytically. Because of the failure of the addition theorem for spherical harmonics for expansion coefficients of helicoidal Bessel beams, an addition theorem for cylindrical Bessel functions is introduced. Meanwhile, an analytical expression for the integral of products including Bessel and associated Legendre functions is applied to eliminate the integration over the polar angle. Note that this multipole expansion may also benefit other scattering methods and expansions of incident waves, for instance, partial-wave series solutions.

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1. Introduction

An ideal Bessel beam is an exact solution of the Helmholtz equation in cylindrical coordinates. A helicoidal Bessel beam possesses an axial null and has an azimuthal phase gradient, which is significantly different from the ordinary (zeroth-order) Bessel beam. The exact series solutions for acoustic scattering from both ordinary[1] and helicoidal[2] Bessel beams with an on-axis incidence were derived by Marston using a superposition of plane waves. It is noteworthy that the off-axis scattering of Bessel beams from a rigid sphere has been investigated by using numerical quadrature to compute the corresponding beam-shape coefficients.[3,4] In addition, the on-axial[5,6] and off-axial[7] acoustic radiation force properties for spherical shapes in Bessel beams were studied using the partial-wave series expansion method. The geometrical interpretation of negative radiation forces in ideal Bessel beams was also discussed and it was found that negative forces only occur when the scattering into the backward hemisphere is suppressed relative to the scattering into the forward hemisphere.[8] It is also of interest to compute the scattering when studying the radiation torque[9–11] caused by Bessel beams. The special features of acoustic radiation forces and torques from Bessel beams may help in the design of devices for manipulating particles and larger objects.

The T-matrix method, one kind of semi-analytical and semi-numerical approach, performs very well for scattering from both spherical and aspherical[12,13] obstacles, which could cover the shortage of series solutions for scattering from aspherical shapes in some ways. Note that the T-matrix method has been demonstrated as an effective tool to calculate acoustic scattering of Bessel beams from complicated shapes.[14,15] Under the structure of the T-matrix method, the incident waves need to be expanded in a proper scalar basis function. For the ordinary Bessel beam, the expansion coefficients of the incident wave were derived on the basis of spherical harmonics

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according to the addition theorem. However, when it comes to the helicoidal Bessel beams, the addition expression of Legendre polynomial fails to expand the incident waves because of the term containing the associated Legendre function $P_n^m(\cos\gamma)$, where γ denotes the angle between the incident and scattered directions and m is an integer with $m \geq 1$. Under this circumstance, a multipole expansion method for acoustical Bessel beams of arbitrary orders is derived to facilitate the application of the T-matrix method for scattering from not only the ordinary Bessel beam, but also the helicoidal beams. It should be noted that the present multipole expansion of Bessel beams could also benefit other methods for more general cases, for example, the series solutions for spheres with off-axis incidence.

2. Addition theory for ordinary Bessel beam

The T-matrix method is effective for acoustic scattering problems in the case where an arbitrary incident wave can be expanded on a proper scalar basis function. When the ordinary Bessel beam is considered, the incident wave can be expressed in spherical coordinates as follows:[5]

$$\psi_{OBB}(r,\gamma) = \psi_0 \sum_{n=0}^{\infty} i^n \times (2n+1) j_n(kr) P_n(\cos\gamma) P_n(\cos\beta), \tag{1}$$

where β is the half-cone angle of Bessel beam and γ is the angle between the incident and scattered waves. The scalar basis functions involve the product of spherical harmonics $Y_{nm}(\theta,\phi)$ and spherical Bessel functions of the first kind $j_n(kr)$, written as

$$\psi_{nm} = Y_{nm}(\theta,\phi) \times j_n(kr), \tag{2}$$

where $Y_{nm}(\theta,\phi) = \xi_{nm} P_n^m(\cos\theta) e^{im\phi}$ is the normalized spherical harmonics of the indicated angular arguments with the normalization coefficients $\xi_{nm} = [(2n+1)(n-m)!]^{1/2} \times [4\pi(n+m)!]^{-1/2}$. It should be noted that for the outgoing scattered waves, the Bessel function needs to be replaced by the spherical Hankle function of the first kind $h_n^{(1)}(kr)$.

To expand the incident Bessel beam of zeroth-order on the basis function, the addition theorem for spherical harmonics is applied, written as

$$P_n(\cos\gamma) = \frac{4\pi}{2n+1} \sum_{m=-n}^{m=n} Y_{nm}(\theta,\phi) Y_{nm}(\theta',\phi'), \tag{3}$$

where (θ,ϕ) is the observation direction and (θ',ϕ') denotes the incident direction. Inserting Eqs. (2) and (3) into Eq. (1), one obtains

$$\psi_{OBB}(r,\gamma) = \sum_{nm} a_{nm} \times \psi_{nm}, \tag{4}$$

where a_{nm} is the expansion coefficients of zeroth-order Bessel beam with the explicit expression given by Gong and co-workers.[14,15] It is worthwhile to note that for helicoidal Bessel beams, the Legendre polynomial $P_n(\cos\gamma)$ would be replaced by associated Legendre functions $P_n^m(\cos\gamma)$ with the addition of an azimuthal phase term of $e^{im\phi}$. Although the associated Legendre functions can be described by the Legendre polynomial using recursion and derivative relationships, the authors could not find an available method to obtain the expansion coefficients of helicoidal Bessel beams using the addition theorem for spherical harmonics. Consequently, a multipole expansion method is introduced to obtain the incident coefficients using the addition theorem for the Bessel functions.

3. Multipole expansion of helicoidal Bessel beams

Consider an arbitrary object under the illumination of a helicoidal Bessel beam with its origin O_{HBB} located in an arbitrary location (x_0, y_0, z_0) in the $Oxyz$ coordinates system, as described by Fig. 1. For an acoustical Bessel beam of arbitrary order, the expression of incident wave could be written as

$$\psi_{ABB} = \psi_0 e^{-i\omega t} i^n e^{ik_z(z-z_0)} \times J_n(k_r R') e^{in\phi'}, \tag{5}$$

where ψ_0 determines the beam amplitude, $k_r = k\sin\beta$ and $k_z = k\cos\beta$ denote the radial and axial wavenumber components, and $R' = \sqrt{(x-x_0)^2 + (y-y_0)^2}$ and $\phi' = \tan^{-1}[(y-y_0)/(x-x_0)]$. Since all the fields have the same time-dependence $e^{-i\omega t}$, it will be omitted throughout. Other than the superposition of plane waves,[1,2] the

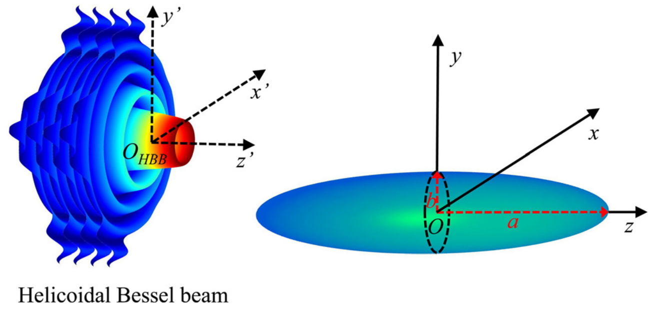

Fig. 1. (Color online) Schematic of an arbitrary object (taking a spheroid as an example) illuminated by a helicoidal Bessel beam of arbitrary orders with its origin O_{HBB} located in arbitrary location (x_0, y_0, z_0) in the $Oxyz$ coordinates system.

direct integration will be implemented in the following to compute the incident coefficients. According to the ideal spherical harmonics expansion,[16] the expansion coefficients of the incident wave could be obtained as

$$\psi_{ABB}(r,\theta,\phi) = \psi_0 \sum_{n'=0}^{\infty} \sum_{m'=-n'}^{n'} a_{n'm'} \times j_{n'}(kr) Y_{n'm'}(\theta,\phi), \tag{6}$$

$$a_{n'm'} = \frac{1}{\psi_0 j_{n'}(kr)} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \psi_{ABB} Y^*_{n'm'}(\theta,\phi) \sin\theta \mathrm{d}\theta \mathrm{d}\phi, \tag{7}$$

where $Y^*_{n'm'}(\theta,\phi)$ is the complex conjugation of normalized spherical harmonics of the indicated angular arguments. Substituting Eq. (5) into Eq. (7), the expansion coefficients still could not be obtained through the double integral over (θ,ϕ). In this circumstance, the addition theorem for the Bessel functions[17] is here subtly introduced to expand the cylindrical Bessel function into the product of two cylindrical Bessel functions with global coordinates (x_0, y_0, z_0) and (x, y, z), and one obtains

$$J_n(k_r R') e^{in\phi'} = \sum_{m=-\infty}^{\infty} J_m(\sigma_0) J_{m+n}(\sigma) e^{i(n+m)\phi} e^{-im\phi_0}, \tag{8}$$

where $\sigma_0 = k_r R_0$, $R_0 = \sqrt{x_0^2 + y_0^2}$, $\phi_0 = \tan^{-1}(y_0/x_0)$, $\sigma = k_r R$, $R = \sqrt{x^2+y^2}$, and $\phi = \tan^{-1}(y/x)$. Inserting Eqs. (5) and (8) into Eq. (7), the expansion coefficients can be rearranged and the exact integral over ϕ is carried out as follows:

$$a_{n'm'} = \frac{i^n e^{ik_z z_0}}{j_{n'}(kr)} \xi_{n'm'} \int_{\theta=0}^{\pi} e^{ik_z z} P_{n'}^{m'}(\cos\theta) \sin\theta \mathrm{d}\theta \times \left[\int_{\phi=0}^{2\pi} \sum_{m=-\infty}^{\infty} J_m(\sigma_0) J_{m+n}(\sigma) e^{i(n+m)\phi} e^{-im\phi_0} e^{-im'\phi} \mathrm{d}\phi \right]. \tag{9}$$

By observation of Eq. (9), only $n + m - m' = 0$ (i.e., $m = m' - n$) will survive through the integral over ϕ and thus $a_{n'm'}$ can be written in the form of a single integral over θ

$$a_{n'm'} = \frac{i^n e^{ik_z z_0}}{j_{n'}(kr)} \xi_{n'm'} \int_{\theta=0}^{\pi} e^{ik_z z} P_{n'}^{m'}(\cos\theta) \sin\theta \mathrm{d}\theta \times 2\pi J_{m'-n}(\sigma_0) J_{m'}(\sigma) e^{-i(m'-n)\phi_0}. \tag{10}$$

To facilitate the implementation of integral operation over θ analytically, an exact solution to the integral on the hybrid product including the associated Legendre, Bessel, and exponential functions in spherical coordinates was verified rigorously by Neves *et al.*[18] as follows:

$$\int_{\theta=0}^{\pi} \sin\theta P_n^m(\cos\theta) e^{ikr\cos\beta\cos\theta} J_m(kr\sin\beta\sin\theta) \mathrm{d}\theta = 2i^{n-m} P_n^m(\cos\beta) j_n(kr). \tag{11}$$

Note that for the expansion coefficients in Eq. (10), the parameters in cylindrical coordinates could be expressed in spherical coordinates, such that $e^{ik_z z} = e^{ikr\cos\beta\cos\theta}$ and $J_{m'}(\sigma) = J_{m'}(kr\sin\beta\sin\theta)$. The expansion coefficients of Bessel beams with arbitrary orders could be derived analytically immediately after substituting Eq. (11) into Eq. (10),

$$a_{n'm'} = 4\pi \xi_{n'm'} i^{n'-m'+n} P_{n'}^{m'}(\cos\beta) \times e^{-ik_z z_0} J_{m'-n}(\sigma_0) e^{-i(m'-n)\phi_0}. \tag{12}$$

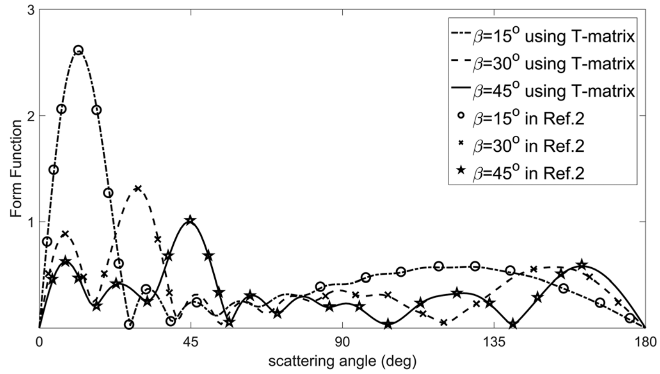

Fig. 2. The form function modulus for scattering by a rigid sphere using the T-matrix method (lines). The parameters selected in this case are all the same as those of Fig. 1 in Ref. 2.

Note that the spherical Bessel function $j_{n'}(kr)$ is canceled by the integration results.

4. Verification and discussions

To verify the correctness of the multipole expansion of acoustical Bessel beams of arbitrary orders (including ordinary and helicoidal Bessel beams), the T-matrix method is applied here for acoustic scattering from a helicoidal Bessel beam for the on-axis case (the origins of the beam O_{HBB} and target O coincide). First, the unknown coefficients of the scattered fields f_{nm} could be computed by multiplying the multipole expansion coefficients of arbitrary-order Bessel beam $a_{n'm'}$ in Eq. (12) by the transition matrix of a considered obstacle $T_{nm,n'm'}$, i.e., $f_{nm} = \Sigma_{n'm'} T_{nm,n'm'} a_{n'm'}$ [see Eq. (13) in Ref. 13 in matrix notation]. Then the scattered scalar basis functions are easily obtained by replacing the Bessel function with spherical Hankel function of the first kind in Eq. (2). Finally, the scattered field could be calculated immediately by multiplying the computed coefficients of the scattered fields by the scattered basis functions.

For brevity, only the scattering form function modulus by a rigid sphere placed in the first-order Bessel beam with different half-cone angle β is presented below, as shown in Fig. 2. The curves represent the form function modulus calculated by the T-matrix method with half-cone angle $\beta = 15^\circ$ (dotted line), $\beta = 30^\circ$ (dashed line), and $\beta = 45^\circ$ (solid line), respectively. While the reference results for these cases, denoted by different kinds of scatters, were extracted from the data using an exact series solution by Marston[2] (given in Fig. 1 of Ref. 2). By comparison, all the results obtained by the T-matrix method using the derived multipole expansion agree very well with those from the partial wave series solution using superposition of plane waves. Moreover, computations using the multipole expansion for scattering by rigid spheroids were also conducted with the T-matrix method (not given for brevity), and the results again agree well with results obtained using a modal matching method.[19]

Note that various methods for acoustic scattering have their own merits. The exact series solutions perform well for scattering from spheres at extremely high frequency. However, the series solutions using spherical harmonics seem to have limitations for aspherical objects with a large aspect ratio due to the ill-condition during matrix inversion procedures. The T-matrix method has advantages for scattering by aspherical obtacles.[12–15] Unfortunately, this method has limitations at high frequency. As a consequence, the exact series solution and the T-matrix method complement each other and provide two powerful tools to investigate the scattering from ordinary and helicoidal Bessel beams. The idea of multipole expansion for acoustic beams may benefit other numerical methods to study more complicated waves (compared with ordinary plane waves), such as smoothed particle hydrodynamics[20] and modified finite element methods.[21–23] The present multipole expansion method is also applicable for scattering from electromagnetic Bessel beams[24] using the T-matrix method.[25] Furthermore, it is noteworthy that the present method could be extended for cases of the off-axis incidence by changing the origin of the incident beam without any extra computation trouble.

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